A geometric formulation to solve eddy current problems in thin conductors of arbitrary topology on general meshes

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We present a novel technique to solve eddy current problems in thin conductors of arbitrary topology by a geometric formulation based on a magnetic scalar potential. The formulation is suitable for an arbitrary polyhedral mesh. A general and fast algorithm is introduced for the topological pre-processing required when the conducting domain is not topologically trivial. Finally, a critical comparison between surface integral and differential formulations is performed.

Index Terms—**eddy currents, discrete geometric approach (DGA), thin conductor, cohomology**

I. INTRODUCTION

THE SOLUTION of eddy current problems in thin conducting structures, such as thin shells (shields), have been already addressed both with differential formulations [1]-[4] and surface integral formulations [5]-[7].

This contribution presents for the first time an efficient geometric formulation, based on the magnetic scalar potential, suitable for general polyhedral meshes.

As far as we know, there are no general and fast algorithms in literature to address the topological pre-processing required when the thin conductor is not topologically trivial. This paper fills this gap by introducing a fast algorithm which exhibits linear complexity on average. Finally, a critical comparison between surface integral and differential formulations is performed. Differential formulations, in fact, lead to a system that may be solved with standard algebraic multigrid solvers, given that it contains only scalar unknowns.

II. EDDY CURRENT PROBLEMS IN THIN CONDUCTORS

Let us consider a polyhedral mesh that covers the computational domain where the eddy current problem has to be solved (a simply connected subset of the three-dimensional Euclidean space). The thin conductor is represented in this mesh by the discrete surface $\mathcal{M}$ formed by the union of polygonal faces.

The nodes belonging to the interior of $\mathcal{M}$ are doubled in such a way that a discontinuity of the electric scalar potential is allowed through the thin layer, see Fig. 1a. Each pair of nodes on $\mathcal{M}$, such as $\{n_i, n_j\}$ in Fig. 1a, possesses the same coordinates but they are considered as two different elements of the resulting cell complex $\mathcal{K}$.

The oriented geometrical elements of $\mathcal{K}$ are nodes n, edges e, faces f and volumes v. The topology of $\mathcal{K}$ is encoded in the incidence matrices $\mathbf{G}$ between the pairs e and n, $\mathbf{C}$ between f and e and $\mathbf{D}$ between v and f. Next, a dual barycentric complex $\tilde{\mathcal{K}}$ is constructed from $\mathcal{K}$ by using the standard barycentric subdivision yielding dual volumes $\tilde{v}$, dual faces $\tilde{f}$, dual edges $\tilde{e}$ and dual nodes $\tilde{n}$ which are in a one to one correspondence (duality) with the geometrical elements n, e, f and v of $\mathcal{K}$, respectively. Thanks to the duality, the incidence matrices of
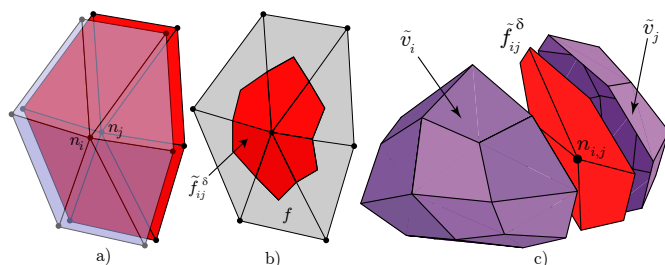

Fig. 1. a) Nodes belonging to the thin layer $\mathcal{M}$ are doubled. In the thin conductor model, the nodes of a pair share exactly the same coordinates. In this picture the pairs of nodes, as $\{n_i, n_j\}$, are displayed one apart from the other only for clarity. b) The additional dual face $\tilde{f}_{ij}^\delta$, dual to the additional edge from n_i to n_j. c) Exploded view of the two dual volumes $\tilde{v}_i$, $\tilde{v}_j$ dual to the pair of nodes $\{n_i, n_j\} \in \mathcal{M}$. The two dual volumes have the additional dual face $\tilde{f}_{ij}^\delta$ in their boundaries, thus the flux on this face has to be considered when enforcing the Gauss law.

$\tilde{\mathcal{K}}$ are deduced from those of $\mathcal{K}$ as: $\tilde{\mathbf{G}} = \mathbf{D}^T$, $\tilde{\mathbf{C}} = \mathbf{C}^T$ and $\tilde{\mathbf{D}} = -\mathbf{G}^T$.

Let us also add to the complex $\mathcal{K}$ the edges, faces and prismatic elements that restitch the complex. Let us call the resulting complex as $\mathcal{K}^\delta$. In particular, we need the incidence matrix $\mathbf{G}^\delta$ between the additional edges and the doubled nodes pairs (as n_i and n_j) and the incidence matrix $\mathbf{C}^{\delta,T}$ between the additional dual faces and additional dual edges. We note that the former incidence matrix can be easily found given that it is the incidence matrix of primal edges and primal nodes of $\mathcal{M}$ before doubling the nodes. These additional edges give rise to additional dual faces (as $\tilde{f}_{ij}^\delta$), see Fig. 1b.

III. THE NOVEL FORMULATION

Let us assume that the thin layer has a small thickness of δ and its electrical resistivity ρ_δ and magnetic permeability μ_δ are piecewise uniform in each mesh element.

Let us introduce $\mathbf{F} = \mathbf{G}\Omega + \mathbf{T}_s + \mathbf{\Pi}\mathbf{i}$ for the edges not inside the shell, whereas $\mathbf{F}^\delta = \mathbf{G}^\delta\Omega + \mathbf{T}^\delta$ for the additional edges inside the shell. Ω and $\mathbf{T}^\delta$ are two unknown degrees of freedom (DoFs) vectors. $\mathbf{\Pi}$ stores in its columns the $H^1(\mathcal{K})$ cohomology generators [9] of the insulating region and $\mathbf{i}$ is the array of unknown independent currents flowing in $\mathcal{M}$

[9]. The full paper will contain the pseudo-code of the novel algorithm to efficiently obtain such a cohomology basis. Then, we introduce $\mathbf{T}_s$, such that $\mathbf{C}\,\mathbf{T}_s = \mathbf{I}_s$, where $\mathbf{I}_s$ is the known source current on mesh faces. This can be found in linear time average complexity by running the Extended Spanning Tree Technique (ESTT) [8].

Le us introduce the constitutive relationships $\tilde{\mathbf{\Phi}} = \mathbf{M}_\mu \mathbf{F}$ and $\tilde{\mathbf{\Phi}}^\delta = \mathbf{M}_\mu^\delta \mathbf{F}^\delta$, where $\tilde{\mathbf{\Phi}}$ denotes the magnetic flux on the standard dual faces, $\tilde{\mathbf{\Phi}}^\delta$ denotes the magnetic flux on the additional dual faces, as $\tilde{f}_{ij}^\delta$ in Fig. 1b, and $\mathbf{M}_\mu^\delta$ is a diagonal matrix whose construction is straightforward and will be described in detail in the full paper.

The discrete Gauss law $\mathbf{G}^T\tilde{\mathbf{\Phi}} + \mathbf{G}^{\delta T}\tilde{\mathbf{\Phi}}^\delta = \mathbf{0}$ enforced on the boundary of all dual volumes (see Fig. 1c), yields

$$\mathbf{G}^T\mathbf{M}_\mu\mathbf{G}\mathbf{\Omega} + \mathbf{G}^T\mathbf{M}_\mu\mathbf{\Pi}\,\mathbf{i} + \mathbf{G}^{\delta T}\tilde{\mathbf{\Phi}}^\delta = -\mathbf{G}^T\mathbf{M}_\mu\mathbf{T}_s, \quad (1)$$

where $\tilde{\mathbf{\Phi}}^\delta$ can be expressed as

$$\tilde{\mathbf{\Phi}}^\delta = \mathbf{M}_\mu^\delta(\mathbf{G}^\delta\mathbf{\Omega} + \mathbf{T}^\delta). \quad (2)$$

Equation (1) has to be complemented with the discrete Faraday's law $\mathbf{C}^{\delta T}\tilde{\mathbf{U}}^\delta + i\omega\tilde{\mathbf{\Phi}}^\delta = \mathbf{0}$ on dual faces in the interior of the thin conductor, where $\tilde{\mathbf{U}}^\delta$ are the electro-motive forces on the additional dual edges in the boundary of the additional dual faces. By defining $\mathbf{I}^\delta$ as the per-unit-length current on primal additional primal faces inside $\mathcal{M}$, the Ohm's constitutive relationship in $\mathcal{M}$ is $\tilde{\mathbf{U}}^\delta = \mathbf{M}_\rho^\delta\mathbf{I}^\delta$.

Therefore, Faraday's law can be written as

$$\mathbf{C}^{\delta T}\mathbf{M}_\rho^\delta\mathbf{C}^\delta\mathbf{T}^\delta + i\omega\tilde{\mathbf{\Phi}}^\delta = \mathbf{0}. \quad (3)$$

Then, the final system becomes

$$\mathbf{K}_\mu\mathbf{\Omega} + \mathbf{G}^{\delta T}\mathbf{M}^\delta\mathbf{T}^\delta + \mathbf{G}^T\mathbf{M}_\mu\mathbf{\Pi}\,\mathbf{i} = -\mathbf{G}^T\mathbf{M}_\mu\mathbf{T}_s$$

$$\mathbf{M}^\delta\mathbf{G}^\delta\mathbf{\Omega} + (\mathbf{K}_\rho^\delta + \mathbf{M}^\delta)\mathbf{T}^\delta + \mathbf{K}_\rho^\delta\mathbf{\Pi}\,\mathbf{i} = \mathbf{0} \quad (4)$$

$$\mathbf{\Pi}^T\mathbf{M}_\mu\mathbf{G}\mathbf{\Omega} + \mathbf{\Pi}^T\mathbf{K}_\rho^\delta\mathbf{T}^\delta + \mathbf{\Pi}^T\mathbf{K}_\rho^\delta\mathbf{\Pi}\,\mathbf{i} = -\mathbf{\Pi}^T\mathbf{M}_\mu\mathbf{T}_s$$

where

$$\mathbf{K}_\mu = \mathbf{G}^T\mathbf{M}_\mu\mathbf{G} + \mathbf{G}^{\delta T}\mathbf{M}^\delta\mathbf{G}^\delta \quad (5)$$

$$\mathbf{K}_\rho^\delta = \frac{1}{i\omega}\mathbf{C}^{\delta T}\mathbf{M}_\rho^\delta\mathbf{C}^\delta \quad (6)$$

The third equation in (4) are the non-local Faraday's laws [9] written on the support of the cohomology generator.

IV. Numerical results

The proposed approach has been applied to calculate the currents induced in a torus of infinitesimal thickness subject to a uniform sinusoidal magnetic field. The numerical domain is covered by a simplicial mesh consisting of 115189 elements and 19764 nodes (22202 nodes after cracking), see Fig. 2. The total number of DoFs is 24641 (22201+ 2438 additional DoFs due to the thin layers + 2 cohomology generators). The solution, shown in Fig. 3, is in excellent agreement with the one obtained with the boundary integral formulation presented in [7]. The full paper will contain a more extensive and quantitative comparison between the two methods.
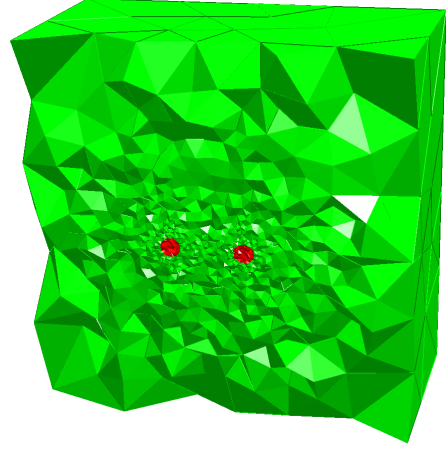
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Fig. 2. Cut-view of the 3-D numerical domain (bounding box: $x, y, z = \pm 10m$) covered by 115189 tetrahedra; in red the elements inside the torus.
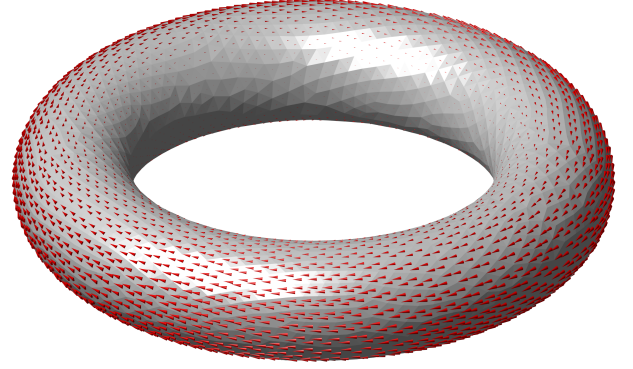

Fig. 3. Eddy currents induced in a torus (major radius $R = 2.0m$, minor radius $a = 0.5m$) of infinitesimal thickness subject to a uniform sinusoidal magnetic field ($B = 1T$, $f = 50Hz$). The torus is covered by a surface mesh consisting of 4876 triangles, 7314 edges, 2438 nodes. Red cones: real part of the current density (a.u.).

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